

Statistics Workshop Part II

Markov Chains

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A **random variable** X is a function $X : S \rightarrow \mathbb{R}$, i.e., each outcome in the sample space is associated with a numerical value.

Example 1: $X =$ number of heads in three coin tosses

| | | | | |
|--------|---------------|---------------|---------------|---------------|
| | | HHT | TTH | |
| S | HHH | HTH | THT | TTT |
| | | THH | HTT | |
| | | | | |
| x | 3 | 2 | 1 | 0 |
| | | | | |
| $p(x)$ | $\frac{1}{8}$ | $\frac{3}{8}$ | $\frac{3}{8}$ | $\frac{1}{8}$ |

A **stochastic process** $\{X_t\}$ is a random variable observed over time.

- **Discrete** or **continuous** time processes.
- A **Markov process** is a process where the next state of the system depends only on the current state (or more generally, on a finite number of immediate past states)
- A discrete Markov process is called a **Markov chain**

Ingredients for Markov Chain

- $X_0, X_1, X_2, \dots, X_n, \dots$ describes the **state** of the chain
- **State space** = $\{1, 2, 3, \dots, N\}$
- (Stationary) **transition probabilities**
$$p_{i,j} = P(X_{n+1} = j | X_n = i)$$
- Graphical display - states are nodes, transitions are edges
- **Transition matrix** $P = (p_{i,j})_{i,j=1}^N$. Note that all rows in P have to add to 1, i.e., P is a **stochastic** matrix.

Example 2: A gambler plays on one of four slot machines, each of which pays off a reward with probability $1/10$. If the player wins on any machine, he continues to play the same machine; otherwise, he switches at random to one of the adjacent machines (where machine 1 and machine 4 are considered adjacent).

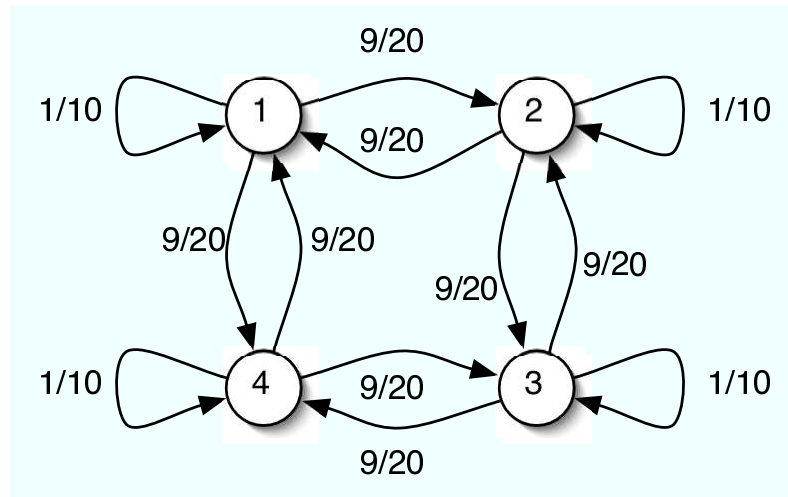
$$S = \{1, 2, 3, 4\}$$

$$P(\text{stay}) = p_{i,i} = \mathbf{1/10} \text{ for all } i = 1, \dots, 4$$

$$P(\text{switch to adjacent}) = p_{i,i+1(\text{mod}4)} = p_{i,i-1(\text{mod}4)} = (9/10)(1/2) = \mathbf{9/20}$$

Transition Matrix and Graphical Representation

$$P = \begin{pmatrix} \frac{1}{10} & \frac{9}{20} & 0 & \frac{9}{20} \\ \frac{9}{20} & \frac{1}{10} & \frac{9}{20} & 0 \\ 0 & \frac{9}{20} & \frac{1}{10} & \frac{9}{20} \\ \frac{9}{20} & 0 & \frac{9}{20} & \frac{1}{10} \end{pmatrix}$$



Questions

- What is the probability for a specific **path**, i.e., sequence of particular states visited?
- What is the probability for moving from state i to j in n steps?
- What is the probability to be in state i at time n ?
- What is the percentage of time that state i occurs in the long run?

For **Question 1**, look at the sequence of states visited
1, 2, 3, 3, 4, 1, 4

We usually represent such a sequence in this form:

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 3 \rightarrow 4 \rightarrow 1 \rightarrow 4$$

$$P(1 \rightarrow 2 \rightarrow 3 \rightarrow 3 \rightarrow 4 \rightarrow 1 \rightarrow 4)$$

$$= P(X_1 = 2 | X_0 = 1) P(X_2 = 3 | X_1 = 2) P(X_3 = 3 | X_2 = 3)$$

$$\cdot P(X_4 = 4 | X_3 = 3) P(X_5 = 1 | X_4 = 4) P(X_6 = 4 | X_5 = 1)$$

$$= p_{1,2} \cdot p_{2,3} \cdot p_{3,3} \cdot p_{3,4} \cdot p_{4,1} \cdot p_{1,4}$$

For **Question 2**, look at $P(X_2 = 1 | X_0 = 1) = p_{1,1}^{(2)}$

| | | | | | | |
|----------------|---|---|---|---|---|-------------------------|
| Possibilities: | 1 | → | 1 | → | 1 | $p_{1,1} \cdot p_{1,1}$ |
| | 1 | → | 2 | → | 1 | $p_{1,2} \cdot p_{2,1}$ |
| | 1 | → | 3 | → | 1 | $p_{1,3} \cdot p_{3,1}$ |
| | 1 | → | 4 | → | 1 | $p_{1,4} \cdot p_{4,1}$ |

$$\begin{aligned} \Rightarrow p_{1,1}^{(2)} &= p_{1,1} \cdot p_{1,1} + p_{1,2} \cdot p_{2,1} + p_{1,3} \cdot p_{3,1} + p_{1,4} \cdot p_{4,1} \\ &= \left(\frac{1}{10}\right)^2 + \left(\frac{9}{20}\right)^2 + 0 + \left(\frac{9}{20}\right)^2 \approx \mathbf{.415} \end{aligned}$$

Note: $p_{1,1}^{(2)} = \sum_{l=1}^4 p_{1,l} \cdot p_{l,1} = (P \cdot P)_{1,1} = (P^2)_{1,1}$

$$\begin{pmatrix} \mathbf{p}_{1,1} & \mathbf{p}_{1,2} & \mathbf{p}_{1,3} & \mathbf{p}_{1,4} \\ p_{2,1} & p_{2,2} & p_{2,3} & p_{2,4} \\ p_{3,1} & p_{3,2} & p_{3,3} & p_{3,4} \\ p_{4,1} & p_{4,2} & p_{4,3} & p_{4,4} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{p}_{1,1} & p_{1,2} & p_{1,3} & p_{1,4} \\ \mathbf{p}_{2,1} & p_{2,2} & p_{2,3} & p_{2,4} \\ \mathbf{p}_{3,1} & p_{3,2} & p_{3,3} & p_{3,4} \\ \mathbf{p}_{4,1} & p_{4,2} & p_{4,3} & p_{4,4} \end{pmatrix}$$

In general, the matrix P^k contains the probabilities for the **k -step transitions probabilities**

$$p_{i,j}^{(k)} = (P^k)_{i,j}$$

For **Question 3**, we use the Law of Total Probability, conditioning on the initial state. Let

$$\vec{p}^{(n)} = (P(X_n = 1), P(X_n = 2), \dots, P(X_n = N))$$

be the vector of **state probabilities at time n** , and $\vec{p}^{(0)}$ be the vector of **initial state probabilities**. Then

$$P(X_n = i) = \sum_{l=1}^n P(X_n = i | X_0 = l) \cdot P(X_0 = l)$$

and we can compute the vector of state probabilities as

$$\vec{p}^{(n)} = \vec{p}^{(0)} \cdot P^n.$$

$$P^1 = \begin{pmatrix} \frac{1}{10} & \frac{9}{20} & 0 & \frac{9}{20} \\ \frac{9}{20} & \frac{1}{10} & \frac{9}{20} & 0 \\ 0 & \frac{9}{20} & \frac{1}{10} & \frac{9}{20} \\ \frac{9}{20} & 0 & \frac{9}{20} & \frac{1}{10} \end{pmatrix}$$

$$P^5 = \begin{pmatrix} 0.17 & 0.33 & 0.17 & 0.33 \\ 0.33 & 0.17 & 0.33 & 0.17 \\ 0.17 & 0.33 & 0.17 & 0.33 \\ 0.33 & 0.17 & 0.33 & 0.17 \end{pmatrix}$$

$$P^{10} = \begin{pmatrix} 0.28 & 0.22 & 0.28 & 0.22 \\ 0.22 & 0.28 & 0.22 & 0.28 \\ 0.28 & 0.22 & 0.28 & 0.22 \\ 0.22 & 0.28 & 0.22 & 0.28 \end{pmatrix}$$

$$P^{15} = \begin{pmatrix} 0.24 & 0.26 & 0.24 & 0.26 \\ 0.26 & 0.24 & 0.26 & 0.24 \\ 0.24 & 0.26 & 0.24 & 0.26 \\ 0.26 & 0.24 & 0.26 & 0.24 \end{pmatrix}$$

$$P^{20} = \begin{pmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{pmatrix}$$

Example 4: Assume the gambler in Example 3 always starts at slot machine 1. What is the probability that the player will move to slot machine **3** after the 10th game? (Note that $X_i =$ machine played in game $i + 1$.)

$\vec{p}^{(0)} = (1, 0, 0, 0)$. Need to compute $\vec{p}^{(10)} = \vec{p}^{(0)} \cdot P^{10}$.

$$\vec{p}^{(0)} \cdot \begin{pmatrix} 0.28 & 0.22 & 0.28 & 0.22 \\ 0.22 & 0.28 & 0.22 & 0.28 \\ 0.28 & 0.22 & 0.28 & 0.22 \\ 0.22 & 0.28 & 0.22 & 0.28 \end{pmatrix} = (0.28 \ 0.22 \ \mathbf{0.28} \ 0.22)$$

For **Question 4**, we need the concept of a stationary distribution and a regular matrix.

A state probability vector is called a **stationary distribution** π if it satisfies $\pi \cdot \mathbf{P} = \pi$. The transition matrix P is **regular**, if, for some n , the entries of the matrix P^n are all positive (no zeros allowed).

Theorem: If P is a **regular** transition matrix, then the powers P^n approach a matrix all of whose rows are the same, and this row vector is the stationary distribution π .

$$P^1 = \begin{pmatrix} \frac{1}{10} & \frac{9}{20} & 0 & \frac{9}{20} \\ \frac{9}{20} & \frac{1}{10} & \frac{9}{20} & 0 \\ 0 & \frac{9}{20} & \frac{1}{10} & \frac{9}{20} \\ \frac{9}{20} & 0 & \frac{9}{20} & \frac{1}{10} \end{pmatrix}$$

$$P^5 = \begin{pmatrix} 0.17 & 0.33 & 0.17 & 0.33 \\ 0.33 & 0.17 & 0.33 & 0.17 \\ 0.17 & 0.33 & 0.17 & 0.33 \\ 0.33 & 0.17 & 0.33 & 0.17 \end{pmatrix}$$

$$P^{10} = \begin{pmatrix} 0.28 & 0.22 & 0.28 & 0.22 \\ 0.22 & 0.28 & 0.22 & 0.28 \\ 0.28 & 0.22 & 0.28 & 0.22 \\ 0.22 & 0.28 & 0.22 & 0.28 \end{pmatrix}$$

$$P^{15} = \begin{pmatrix} 0.24 & 0.26 & 0.24 & 0.26 \\ 0.26 & 0.24 & 0.26 & 0.24 \\ 0.24 & 0.26 & 0.24 & 0.26 \\ 0.26 & 0.24 & 0.26 & 0.24 \end{pmatrix}$$

$$P^{20} = \begin{pmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{pmatrix}$$

- P is a regular transition matrix (P^5 , for example, has all positive entries) \Rightarrow stationary distribution exists
- P^{20} suggests that $\pi = (1/4, 1/4, 1/4, 1/4)$
- Check: $\pi \cdot P = \pi$?

Note: If no computer is available to compute matrix powers of P , one can solve for $\pi = (\pi_1, \pi_2, \dots, \pi_N)$ by setting up the system of N equations corresponding to $\pi \cdot P = \pi$ and solve for π_i , $i = 1, \dots, N$. Due to redundancy in this system, we also need to use that $\sum_{i=1}^N \pi_i = 1$.

$$(\pi_1 \ \pi_2 \ \pi_3 \ \pi_4) = \begin{pmatrix} \frac{1}{10} & \frac{9}{20} & 0 & \frac{9}{20} \\ \frac{9}{20} & \frac{1}{10} & \frac{9}{20} & 0 \\ 0 & \frac{9}{20} & \frac{1}{10} & \frac{9}{20} \\ \frac{9}{20} & 0 & \frac{9}{20} & \frac{1}{10} \end{pmatrix} \cdot (\pi_1 \ \pi_2 \ \pi_3 \ \pi_4)$$

This gives the following system of equations:

$$\frac{1}{10} \pi_1 + \frac{9}{20} \pi_2 + \frac{9}{20} \pi_4 = \pi_1 \quad (1)$$

$$\frac{9}{20} \pi_1 + \frac{1}{10} \pi_2 + \frac{9}{20} \pi_3 = \pi_2 \quad (2)$$

$$\frac{9}{20} \pi_2 + \frac{1}{10} \pi_3 + \frac{9}{20} \pi_4 = \pi_3 \quad (3)$$

$$\frac{9}{20} \pi_1 + \frac{9}{20} \pi_3 + \frac{1}{10} \pi_4 = \pi_4 \quad (4)$$

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \quad (5)$$

Summary

The stationary distribution can be found in two ways:

1. Use software to compute high powers of the transition matrix. When all rows are identical, read off the stationary distribution. Verify by computing $\pi \cdot P$ and checking that the result equals π .
2. Solve the system of equations given by $\pi \cdot P = \pi$,
 $\sum_{i=1}^N \pi_i = 1$ for π .

The stationary distribution gives the **long term probabilities** for the Markov chain to be in a given state, and does NOT depend on the initial distribution $\vec{p}^{(0)}$ (see exercises).